THE INSCRIPTION OF REGULAR POLYGONS.

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CHAPTER VI.

[Concluded from the January Number.]

IV. Let $n=mr$, where, n being odd, m and r are both odd. Then $A_s-A_{m-s}-A_{m+s}+A_{2m-s}+A_{2m+s}-\ldots.$

$$+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m-s}+(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m+s}=0\ldots.(6).$$

Proof: $A_m-A_{2m}+A_{3m}-A_{4m}+\ldots.-(-1)^{\frac{r-1}{2}}A_{\frac{r-1}{2}m}=1$, being chords of the regular r-gon. Multiplying both sides by A_s, we find

$A_s=A_{m-s}+A_{m+s}-A_{2m-s}-A_{2m+s}+\ldots.$ By the method so often used we may prove that the r chords of (6) $A_s, -A_{m-s}, -A_{m+s}, A_{2m-s}$, etc., are the roots of $x^r-rx^{r-2}+\frac{r(r-3)}{1.2}x^{r-4}-\frac{r(r-4)(r-5)}{1.2.3}x^{r-6}+\ \ldots$

$$+(-1)^p\frac{r(r-p-1)(r-p-2)\ldots.(r-2p+1)}{1.2.3\ldots.p}x^{r-2p}+\ldots.\pm rx-A_{rs}=0\ldots.(7).$$

This may be proved directly from the trigonometric formula:

$$2\cos r\theta=2^r\cos^r\theta-2^{r-2}r\cos^{r-2}\theta+2^{r-4}\frac{r(r-3)}{1.2}\cos^{r-4}\theta$$

$$-\ldots.+(-1)^p2^{r-2p}\frac{r(r-p-1)(r-p-2)\ldots.(r-2p+1)}{1.2.3\ldots.p}\cos^{r-2p}\theta+\ldots.$$ from

which it follows that $A_s=2\cos\frac{s\pi}{mr}$ is a root of equation (7).

Also $-A_{m-s}=-2\cos\frac{(m-s)\pi}{mr}$ is a root of (7); for, θ then being $\frac{(m-s)\pi}{mr}$,

$2\cos r\theta=2\cos\frac{(m-s)\pi}{m}=-2\cos\frac{s\pi}{m}=-A_{rs}$. Similarly, $-A_{m+s}$, A_{2m-s}, A_{2m+s}, etc , are roots of (7). In another article I give a direct geometric-algebraic proof based upon the principles given above and the theory of symmetric functions.

If m is prime to r, one chord of each of the above groups (6) of r chords each is a root of our general equation (4) for a regular m-gon. For one

and only one of the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$, ... $\frac{r-1}{2}m-s$, $\frac{r-1}{2}m+s$ is always divisible by r, as is seen by writing them in the equivalent form: $rm-s$, $m-s$, $(r-1)\,m-s$, $2m-s$, $(r-2)\,m-s$, ... $\frac{r-1}{2}\,m-s$, $\frac{r+1}{2}\,m-s$, respectively. The remaining $r-1$ chords will be determined by equations whose degrees are given by the prime factors of $r-1$, — a chain of equations in which the coefficients of any one are linear functions of the roots of the preceding and of the roots of (4) for the m-gon. Hence, if the $\frac{m-1}{2}$ chords of the regular m-gon be found, we determine all the chords of the regular rm-gon by solving a series of equations whose degrees are the prime factors of $r-1$.

However, if m is divisible by r, the r chords in any of the above groups are all, or not one of them, roots of (4); for the subscripts s, $m-s$, $m+s$, $2m-s$, $2m+s$, ... are all or not one divisible by r, according as s is or is not divisible by r. Hence, by the grouping of the $\frac{n-1}{2}$ chords of the n-gon into $\frac{m-1}{2}$ groups of r chords each, we can not lower or avoid equations of the rth degree of the form (7). More definitely, if r be a prime number and if m be divisible by r, we must, for any grouping whatever of the chords of the rm-gon, solve one or more equations of degree r.

The regular polygon of mr sides depends for inscription, if m be prime to r, upon the same equations as does the regular m-gon, together with equations whose degrees are the prime factors of $r-1$; but, if m contains as factor the prime number r, upon an equation of degree r and of the form (7), in addition to those required by the regular m-gon.

To inscribe a regular polygon of $n=a^{\alpha}b^{\beta}c^{\gamma}$.... sides, therefore, where a, b, c, ... are *different* prime numbers, it is necessary to solve $\alpha-1$ equations of degree a, $\beta-1$ of degree b, etc., besides equations whose degrees are given as the prime factors of $\frac{a-1}{2}$, $\frac{b-1}{2}$, $\frac{c-1}{2}$, ...

It follows that the regular $(2^x+1)m$-gon depends for inscription upon the same equations as the regular m-gon, provided 2^x+1 be a prime number, and is inscriptible if the latter is. Hence, a regular polygon of $(2^x+1)(2^y+1)(2^z+1)$.... sides, where the factors are *different* prime numbers, is geometrically inscriptible. We thus have Gauss' theorem:

A regular polygon the number of whose sides is a prime number of the form 2^x+1, or the product of two or more different primes of that form, or a power of 2 times such an expression, is geometrically inscriptible; and inversely.

Of the regular polygons with less than 200 sides, 31 are geometrically inscriptible:

3,4,5,6,8,10,12,15,16,17,20,24,30,32,34,40,48,51,60,64,68,80,85,96, 102, 120,128,136,160,170,192;

67 depend for inscription upon *cubics* only: 7,9,13,14,18,19,21,26,27, 28,35,36,37,38,39,42,45,52,54,56,57,63,65,70,72,73,74,76,78,81,84,90, 91, 95, 97, 104,105,108,109,111,112,114,117,119,126,130,133,135,140,144,146,148,152,153, 156,162,163,168,171,180,182,185,189,190,193,194,195;

23 depend upon *quintics* only: 11,22,25,33,41,44,50,55,66,75,82,88, 100,101,110,123,125,132,150,164,165,176,187; 17 depend upon *cubics* and *quintics*: 31,61,62,77,93,99,122,124,143,151,154,155,175,181,183,186,198;

8 depend on equations of 7th degree only: 29,58,87,113,116,145,174,197;

9 depend on equations of 3rd and 7th degrees: 43,49,86,98,127,129,147,172, 196; 2 on equations of 5th and 7th degrees: 71 and 142; the 40 remaining depend on equations of the 11th or higher degrees.

An idea of the comparative infrequency of the geometrically inscriptible regular polygons if we advance to large numbers is found in the fact (as shown by a complete table I have made) that there are only 206 of them with sides less than a million.

In a paper* on *The Number of Inscriptible Regular Polygons*, I proved that between the successive powers of 2 lie 1,2,3,4,5, etc. numbers giving inscriptible regular polygons; thus

[3],4,[5,6],8,[10,12,15],16,[17,20,24,30],32,[34,40,48,51,60],64,....

or generally, n such numbers lie between 2^n and 2^{n+1}, for every value of $n<32$. For $n \geqq 32$ but <128, 31 such numbers lie between 2^n and 2^{n+1}. For values of $n \geqq 128$, there would be 31 such numbers in each interval *if* (as is not yet known) $2^{128}+1$ is a composite number; but $31+l$ such numbers between 2^{128+l} and 2^{129+l}, if it is a prime number, l being <128.

Hence the number of inscriptible regular polygons below 2^x+1 sides, for $x<32$, is $\frac{1}{2}(x-1)(x+2)$; for $x \geqq 32$, but <128, is $(32x-497)$.

In conclusion, I will add a direct geometric proof of our fundamental theorem (5): $A_1-A_2+A_3-A_4+A_5-\ldots\ldots-(-1)^pA_p=1$.

Suppose $A_1-A_2+A_3-\ldots\ldots\pm A_p=x$. $\therefore\ xA_1=A_1(A_1-A_2+A_3-\ldots\ldots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_2-A_1-A_3+A_2+A_4-A_3-A_5+\ldots\ldots\pm A_{p-3}\pm A_{p-1}\pm A_{p-2}\pm A_p\pm A_{p-1}\pm A_p=2+A_1-2(A_1-A_2+A_3-\ldots\ldots\pm A_{p-2}\pm A_{p-1}\pm A_p)=2+A_1-2x$.

$$\therefore\ (x-1)(2+A_1)=0.\quad \text{But } A_1+2 \neq 0.\quad \therefore\ x=1.$$

Errata in October Number:—p. 343, line 20 add *when n is a prime number*; p. 344, line 8 read $A_s^2=2-A_{n-2s}$.

* Bulletin of the New York Mathematical Society, February, 1894.